Applications of Lefschetz numbers in control theory

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Abstract. We develop some applications of techniques of the Lefschetz coincidence theory in control theory. The topics are existence of equilibria and their robustness, controllability and its robustness.

1. Introduction.

The goal of this paper is to provide examples of what Lefschetz coincidence theory can contribute to control theory. We discuss existence of equilibria and their robustness, controllability and its robustness.

We develop some topological techniques, already available in dynamics, in the control theoretic setting. A (discrete) dynamical system on a manifold M is simply a map $f : M \to M$. Then $x \in M$ and $f(x)$ are the current and next states of the system respectively. An equilibrium of the system is a fixed point of f.The problem of detecting equilibria can be treated via the more general Coincidence Problem [**2**, VI.14], [**35**, Ch. 7], [**15**]: "Given two maps $f, g : N \to M$ between two n-dimensional manifolds, what can be said about the coincidence set C of all x such that $f(x) = g(x)$?" Indeed, the equilibrium set $C = \{x \in M : f(x) = x\}$ is the coincidence set of f and the identity map $g : M \to M$. The famous Lefschetz coincidence theorem states that if the Lefschetz number λ_{fg} is not equal to zero then there is at least one coincidence, i.e., $C \neq \varnothing$. Using this and other invariants one can find out whether a dynamical system has an equilibrium or a periodic point.

In case of a *controlled* dynamical system, the next state $f(x, u)$ depends not only on the current state, $x \in M$, but also on the *input*, $u \in U$. A discrete time control system is given by the space of inputs U, the space of states M, the "state-input" space $N = M \times U$, a map $f : N = M \times U \to M$, and the projection $g : N = M \times U \to M$ (in general N is a fiber bundle and $g : N \to M$ is the bundle projection). Then, just as above, the equilibrium set $C = \{x \in M : f(x, u) = x$ for some $u \in U\}$ of the system is the coincidence set of the pair (f, g). However, since the dimensions of N and M are not equal anymore, the Lefschetz *number* is replaced with the Lefschetz *homomorphism* [**31**] which does a better job at detecting coincidences.

Another application of the coincidence theory approach is controllability. A system is called controllable if any state can be reached from any other state,

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i.e., for each pair of states $x, y \in M$ there are inputs $u_0, ..., u_r \in U$ such that $x_1 = f(u_0, x), x_2 = f(u_1, x_1), ..., y = x_{r+1} = f(u_r, x_r)$. Therefore controllability is equivalent to surjectivity of the composition of several iterations of f. On the other hand a map is surjective if it has a coincidence with any constant map.

The state space M is often a manifold, as opposite to a Euclidean space, when it appears in robotics. For example, $M = \mathbf{T}^n = (\mathbf{S}^1)^n$, the n-dimensional torus, is the space of all possible states of a robotic arm with n revolving joints [**27**, p. 1]; or $M = \mathbf{R}^3 \times SO(3)$ is the space of positions of a rigid body [**25**, Chapter 2]. Typically, we have $N = M \times U$. However nontrivial bundles are also common. For example, consider a spherical pendulum with a gas jet control which is always directed in the tangent space. Then its state space is $M = \mathbf{S}^2$, the 2-sphere, while the state-input space N is the tangent bundle $T\mathbf{S}^2$ of $\mathbf{S}^2$, which is an $\mathbf{R}^2$-bundle over M not isomorphic to $M \times \mathbf{R}^2$ [**27**, p. 17]. In spite of the abundance of such examples [**6**], [**25**], [**27**] topological techniques have not thus far found broad applications in control theory. The only recent examples known to the author are [**18**] - [**22**].

The topological approach provides the following advantages. Consider a control system as a triple (M, N, f) of topological spaces M, N and a continuous map f as described above. Since our knowledge of the model is inevitably imprecise, we have to deal with perturbations of the system. As perturbations may be understood as variations of unknown parameters of the system their effect on the behavior of the system is also unknown. However, if the system depends continuously on these parameters, the change of M, N, and f is also continuous. This means that we are to consider spaces homeomorphic to M, N and maps homotopic to f. An appropriate tool to deal with this degree of generality is homology theory. Indeed, the homology groups $H_*(M), H_*(N)$ of M, N and the homology homomorphism $f_* : H_*(N) \to H_*(M)$ of f remain constant under homeomorphisms of M, N and homotopies of f. They can also be rigorously and effectively computed [**26**], [**19**].

Further, the perturbations of f are normally assumed "small" (in particular, this is the basis of the notion of structural stability). However unless actual estimates are available, we don't know how "small" are the perturbations of the real system. Therefore in order to take into account the "worst possible scenario" we consider large, but still continuous, perturbations of the system. As an example, a constant external force, such as gravitation, in any of the above robotic systems may be treated as such a perturbation. Thus the use of homology theory provides answers with a new, for control theory, degree of robustness. Providing results of this nature is the first objective of this paper. We apply Lefschetz coincidence theory to prove existence of equilibria (Theorem 6.1) and controllability (Theorem 7.2) for systems determined by maps homotopic to f.

The second objective of this paper is to study robustness of these properties under arbitrarily small perturbations because sometimes they produce a dramatic change in the properties of the system. This change may be the loss of an equilibrium (Theorem 6.4) or the loss of controllability (Theorem 7.3).

The paper is organized as follows. Some preliminaries from algebraic topology are outlined in the Section 2. In Section 3 we review the classical theory of Lefschetz numbers and show its inadequacy for control theory. In Section 4 we consider the necessary generalization, the Lefschetz homomorphism, of the Lefschetz number and state several relevant results about existence of coincidences. In Section 5 we state some results about removability of coincidences. In Section 6 we provide

sufficient conditions of existence of equilibria of a discrete system and their robustness. In Section 7 we provide sufficient conditions of controllability of a discrete system and its robustness. In Section 8 we discuss how our coincidence results can be applied to existence of equilibria and controllability of continuous time control systems. Notions of control theory are defined as needed, for details see [**27**], [**29**], [**34**].

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2. Preliminaries from algebraic topology.

The terminology we use is standard [**2**]. Suppose N is a topological space and $A \subset N$ is a subspace. The singular *homology group* $H_k(N,A)$ of N relative to A over $\mathbf{Q}$ or any other field is defined as follows. If Δ_k is the standard k-simplex, $k = 0,1,2...$, any map $\sigma : \Delta_k \to N$ is called a singular k-simplex in N. We let $C_k(N,A)$ be the vector space over $\mathbf{Q}$ generated by all singular k-simplices of N whose images are not completely in A. Then the boundary operator $\partial_k : C_k(N,A) \to C_{k-1}(N,A)$ is defined in the natural way and we let $H_k(N,A) = \ker \partial_k / \operatorname{Im} \partial_k$. Further, let $C^k(N,A)$ be the dual of $C_k(N,A)$, i.e., the vector space of all linear functions from $C_k(N,A)$ to $\mathbf{Q}$. Then ∂_k generates the coboundary operator $\partial^k : C^k(N,A) \to C^{k+1}(N,A)$ and we let $H^k(N,A) = \ker \partial^k / \operatorname{Im} \partial^k$ be the *cohomology group* of N relative to A. Also $H_k(N) = H_k(N,\varnothing)$, $H^k(N) = H^k(N,\varnothing)$. If (N,A) is a simplicial complex, its simplicial homology and cohomology is defined in the same way starting with $C_k(N,A)$ generated by all simplices of (N,A). The homology and cohomology groups $H_k(N,A;G)$, $H^k(N,A;G)$ over any group G can be defined in a similar fashion.

Homology and cohomology groups $\{H_k(N,A) : k = 0,1,2...\}$, $\{H^k(N,A) : k = 0,1,2...\}$ over fields are (graded) vector spaces with the following properties. The Betti numbers, $b_k = \dim H_k(N)$, for $k = 0,1,2$, are the numbers of path components, "tunnels", and "voids" of N, respectively. In case of a path connected N, the identities of $H_0(N) = H^0(N) = \mathbf{Q}$ are denoted by 1. If N is contractible, it is *acyclic*, i.e., $H_k(N) = H^k(N) = 0$ for $k > 0$. If N is an n-dimensional simplicial complex, $H_k(N) = 0$ for all $k > n$. If M is a compact connected orientable n-dimensional manifold with boundary ∂M then $H_n(M,\partial M) = H^n(M,\partial M) = \mathbf{Q}$. The identities of these two groups are the *fundamental classes* O_M and $\overline{O}_M$ of M respectively. Further, there is the *Poincaré duality* isomorphism $D_M : H^k(M,\partial M) \to H_{n-k}(M)$ given by the cap product with the fundamental class O_M. The *cap product* is the homomorphism $\frown : H^k(N,A) \otimes H_m(N,A) \to H_{m-k}(N)$ given by $x \frown a = (1 \times x)\Delta a$, where Δ is a diagonal approximation. Then $a \in H_k(N,A)$ and $x \in H^k(N,A)$ are called *dual* if $x \frown a = \langle x,a \rangle = x(a) = 1$. In particular, O_M and $\overline{O}_M$ are dual. By the Künneth Theorem, $H_k(M \times U) = \sum_{i+j=k} H_i(M) \otimes H_j(U)$, $k = 0,1,2,....$

Suppose B is a subspace of the topological space M and $f : N \to M$ is a map, then $f : (N,A) \to (M,B)$ is a *map of pairs* if $f(A) \subset B$. In this case f generates the natural homomorphism from $C_k(N,A)$ to $C_k(M,B)$. This homomorphism generates $f_* : H_k(N,A) \to H_k(M,B)$, the *homology homomorphism* of f, and $f^* : H^k(M,B) \to H^k(N,A)$, the *cohomology homomorphism* of f. Two maps $f,g : (N,A) \to (M,B)$ are called *homotopic* if f can be continuously "deformed" into g, i.e., there is a map $F : [0,1] \times (N,A) \to (M,B)$ such that $F(0,\cdot) = f$ and

$F(1,\cdot) = g$. If f and g are homotopic then $f_* = g_*$. In particular, if f is homotopic to a constant map then f_* is trivial, i.e., $f_* : H_k(N, A) \to H_k(M, B)$ is zero for $k = 1, 2, ...,$ or simply $f_* = 0$. In the case of n-manifolds, the homomorphism $f_* : H_n(N, \partial N) \to H_n(M, \partial M)$ is the multiplication by $\deg f$, the degree of f. The kth homotopy group $\pi_k(N)$ of N is the group of homotopy classes of maps of k-spheres to N.

3. Review of Lefschetz theory.

In this section M and N are orientable compact connected manifolds with boundaries $\partial M, \partial N$, and $\dim M = \dim N = n$.

Consider the Fixed Point Problem: "If $f : M \to M$ is a map, what can be said about the set of points $x \in M$ such that $f(x) = x$?" Applications of fixed point theorems (Kakutani, Banach, etc.) to control problems are abundant, [**1**], [**7**], [**8**], [**16**], [**23**], [**28**]. However the methods we suggest in this paper go far beyond those.

One may associate to f an integer λ_f called the Lefschetz number [**3**]:

$$\lambda_f = \sum_k (-1)^k Trace(f_{*k}),$$

where $f_{*k} : H_k(M) \to H_k(M)$ is induced by f. The *Lefschetz fixed point theorem* states that if $\lambda_f \neq 0$, then f has a fixed point.

The Coincidence Problem is concerned with a similar question about two maps $f, g : N \to M$ and their coincidences $x \in N, f(x) = g(x)$. One of the main tools is the Lefschetz coincidence number λ_{fg} defined similarly to λ_f as the alternating sum of traces of a certain endomorphism on the homology group of M. Algebraically, if $h : E_* \to E_*$ is a (degree 0) endomorphism of a finitely generated graded vector space $E_* = \{E_k\}$, given by $h_k : E_k \to E_k$, then its Lefschetz number is $L(h) = \sum_k (-1)^k Trace(h_k)$. To apply this formula in the topological setting we let $E_* = H_*(M)$, then the Lefschetz number is defined as $\lambda_{fg} = L(g_* D_N f^* D_M^{-1})$, where $D_M : H^k(M, \partial M) \to H_{n-k}(M)$, $D_N : H^k(N, \partial N) \to H_{n-k}(N)$ are the Poincaré duality isomorphisms. Observe that for $f^* : H^k(M, \partial M) \to H^k(N, \partial N)$ to be well defined the map f has to be boundary preserving, $f : (N, \partial N) \to (M, \partial M)$.

A *Lefschetz type coincidence theorem* states that if $\lambda_{fg} \neq 0$ then the pair (f, g) (and any pair homotopic to them) has a coincidence. The converse is false in general. When $\lambda_{fg} = 0$, the maps f, g may have coincidences but under certain circumstances they can be removed by homotopies of f, g [**5**].

Until recently such theorems have been mostly considered in the following two settings. First [**2**, VI.14], [**35**, Ch. 7], $f : N \to M$ is a map between two n-manifolds as above. This way the Lefschetz theorem can be applied to detect equilibria of a dynamical system but it does not apply to an even simplest control system because the dimensions of $N = M \times U$ and M have to be equal. Second [**15**], $f : X \to M$ is a map from an arbitrary topological space X to an open subset of $\mathbf{R}^n$ and all fibers $f^{-1}(y)$ are acyclic. Here the dimensions are also equal in the sense that $H_*(X) = H_*(M)$ (Vietoris Theorem). Thus neither case is broad enough to cover control systems the input spaces U of which have nonzero dimension.

As an example from dynamics, consider the problem of existence of closed orbits of a flow. The flow is given by a map $f : M \times [0, \infty) \to M$ so that the initial position is $f(0, x) = x$ and $f(t, x)$ is the position at time t. Closed orbits correspond to coincidences of f and the projection $p : M \times [0, \infty) \to M$. More generally one

considers $f : M \times X \to M$, where X is a topological space. This situation was studied in [**24**], [**12**], [**13**], [**11**] under the name "parametrized fixed point theory". These results can be applied to detect equilibria (Section 6), but the setting is not general enough to study controllability (Section 7). The author [**30**], [**31**] extended some of the results of [**13**] to the general case of two arbitrary maps $f, g : N \to M$ from an arbitrary topological space to a manifold. The content of these papers is briefly outlined in the next section.

4. Detecting coincidences.

In this section N is an arbitrary topological space, $A \subset N$, M is an orientable compact connected manifold with boundary ∂M, $\dim M = n$, and $f : (N, A) \to (M, \partial M)$, $g : N \to M$ are maps.

The generalization of the Lefschetz number is based on the fact that since the finitely generated graded vector space $E = H_*(M)$ is equipped with the cap product $\frown: E^* \otimes E_* \to E_*$, one can define the Lefschetz class $L(h) \in E_*$ of a graded endomorphism h given by $h_k : E_k \to E_{k+m}$ of any degree m not just of degree 0 as in the classical case.

Definition 4.1. [**31**, Proposition 2.2] If $h : H_k(M) \to H_{k+m}(M)$, $k = 0, 1, 2...$, is a graded homomorphism of degree m then the *Lefschetz class* $L(h) \in H_m(M)$ is defined as

$$L(h) = \sum_k (-1)^{k(k+m)} \sum_j x_j^k \frown h(a_j^k),$$

where $\{a_1^k, ..., a_{m_k}^k\}$ is a basis for $H_k(M)$ and $\{x_1^k, ..., x_{m_k}^k\}$ the corresponding dual basis for $H^k(M)$.

It is easy to see that if the degree m of h is zero, $L(h) = \sum_k (-1)^k Trace(h_k)$.

For a given $z \in H_s(N, A)$, suppose the homomorphism h_{fg}^z is defined as the composition

$$H_i(M) \xrightarrow{D_M^{-1}} H^{n-i}(M, \partial M) \xrightarrow{f^*} H^{n-i}(N, A) \xrightarrow{\frown z} H_{s-n+i}(N) \xrightarrow{g_*} H_{s-n+i}(M),$$

i.e.,

$$h_{fg}^z(x) = g_*((f^* D_M^{-1}(x)) \frown z)).$$

Its degree is $m = s - n$.

Definition 4.2. The *Lefschetz homomorphism* $\Lambda_{fg} : H_s(N, A) \to H_{s-n}(M)$, $k = 0, 1, ...$, of the pair (f, g) is defined by

$$\Lambda_{fg}(z) = L(h_{fg}^z).$$

The degree of the homomorphism h_{fg}^z is zero if $z \in H_n(N, A)$. If, moreover, N is a orientable compact connected manifold of dimension n, we have $H_n(N, \partial N) = \mathbf{Q}$. Its identity is the fundamental class $O_N \in H_n(N, \partial N)$ of N. Since $D_N(x) = x \frown O_N$, we recover the classical *Lefschetz number*, $\lambda_{fg} = \Lambda_{fg}(O_N)$.

Theorem 4.3. [**31**, Theorem 6.1] ***(Existence of coincidences)*** *If $\Lambda_{fg} \neq 0$ then any pair of maps f', g' homotopic to f, g has a coincidence.*

Especially important for the control theory applications are the following two corollaries. They are applied to existence of equilibria (Section 6) and controllability (Section 7) respectively. Observe that the second corollary is about a map of pairs and the first is not.

Corollary 4.4. ***(Existence of fixed points)*** *(cf.* [**13**]*) Let $g : M \times U \to M$ be a map. Given $v \in H_s(U)$, suppose the homomorphism $g_v : H_i(M) \to H_{i+s}(M), i = 0, 1, \ldots,$ of degree s is defined by*

$$g_v(x) = (-1)^{(n-i)s} g_*(x \otimes v),$$

$x \in H_i(M)$. Then, if

$$L(g_v) \neq 0 \text{ for some } v \in H_s(U)$$

then any map $g' : M \times U \to M$ homotopic to g has a fixed point x, $g'(x, u) = x$ for some u.

Proof. Let $(N, A) = (M, \partial M) \times U$ and apply the above theorem to the pair p, g, where $p : (M, \partial M) \times U \to (M, \partial M)$ is the projection. Also according to Corollary 5.7 in [**31**], $\Lambda_{pg}(O_M \otimes v) = L(g_v)$. □

Corollary 4.5. ***(Sufficient condition of surjectivity)*** *If*

$$f_* : H_n(N, A) \to H_n(M, \partial M) = \mathbf{Q} \text{ is nonzero}$$

then any map $f' : (N, A) \to (M, \partial M)$ homotopic to f is onto.

Proof. Apply the theorem to the pair f, c, where c is any constant map (as in Section 5 in [**30**] and Proposition 6.8 in [**31**]). □

In case of manifolds of equal dimensions the condition of this corollary is equivalent to the nonvanishing of the degree $\deg f$ [**2**, p. 186] of f.

5. Removing coincidences.

In this section M is a compact orientable connected manifold with boundary ∂M, $\dim M = n$, N is a manifold, $f, g : N \to M$ are maps.

When $\dim N = \dim M = n > 2$, the vanishing of the Lefschetz number λ_{fg} implies that the coincidence set can be removed by homotopies of f, g [**5**]. If $\dim N = n + m, m > 0$, this is no longer true even if λ_{fg} is replaced with Λ_{fg}. Some progress has been made for $m = 1$. In this case the secondary obstruction to the removability of a coincidence set was considered in [**10**], [**9**], [**17**]. These results can be used to study removability of equilibria when the dimension of the input space is 1. However the conditions on f and g are hard to verify. Necessary conditions of the global removability for arbitrary m were considered in [**14**, Section 5] with N a torus and M a nilmanifold. For some $m > 1$, a partial converse of Theorem 4.3 is provided by the author [**32**]. A version of this theorem is given below.

Suppose F is an isolated subset of the coincidence set of f, g and $f(F) = g(F) = \{x\}$, $x \in M \backslash \partial M$. Let D be a open neighborhood of x such that $D \cap \partial M = \varnothing$. Choose a neighborhood W of F in N with no coincidences such that $f(W) \subset D$ and $g(W) \subset D$. Suppose $V \subset \overline{V} \subset W$ is another neighborhood of F, then there is an open neighborhood $B \subset \overline{B} \subset D$ of x such that $f(W \backslash V) \subset D \backslash B$. Therefore $f : (W, W \backslash V) \to (D, D \backslash B)$ is a map of pairs.

Theorem 5.1. ***(Local removability of coincidences)*** *Suppose the following property is satisfied*

$$(*) \quad H^{k+1}(W, W \backslash V; \pi_k(\mathbf{S}^{n-1})) = 0 \text{ for } k \geq n + 1.$$

Suppose also that

$$f_* : H_n(W, W\backslash V) \to H_n(D, D\backslash B) = \mathbf{Q} \text{ is zero.}$$

Then there is a homotopy of f constant on the compliment of V to a map f' such that the new pair has no coincidences in V.

Since D is arbitrary we can say that the homotopy can be chosen *arbitrarily small.*

Proof. According to the proof of Theorem 2 in [**32**] the coincidence subset F can be removed by a homotopy of f constant on $N\backslash V$ provided the local cohomology index $I^W_{fg}(\tau)$ vanishes. This index is defined as follows. Since $F \subset V$ is the set of all coincidences in W, the map $(f, g) : (W, W\backslash V) \to D^\times = (D \times D, D \times D\backslash d(D))$, where $d(D)$ is the diagonal of $D\times D$, is well defined. Therefore the homomorphisms $(f, g)_* : H_k(W, W\backslash V) \to H_k(D^\times)$ and $(f, g)^* : H^k(D^\times) \to H^k(W, W\backslash V)$ are also well defined. Now let I_{fg} be the homology coincidence homomorphism defined by $I_{fg} = (f, g)_* : H_k(W, W\backslash V) \to H_k(D^\times)$. Let $I^W_{fg}(\tau) = (f, g)^*(\tau) \in H^n(W, W\backslash V)$ be the cohomology coincidence index [**32**, Section 2], where τ is the identity of $H^n(D^\times) = \mathbf{Q}$. By Theorem 6.1 in [**31**], $\Lambda_{fg}(z) = \pi_*(\tau \frown I_{fg}(z))$, where $\pi : D \times D \to D$ is the projection on the first factor. Then, for any $z \in H_n(W, W\backslash V)$

$$\begin{aligned}\Lambda_{fg}(z) &= \pi_*(\tau \frown (f, g)_*(z)) = \pi_*(f, g)_*((f, g)^*(\tau) \frown z) \\ &= \langle (f, g)^*(\tau), z\rangle = \left\langle I^W_{fg}(\tau), z\right\rangle .\end{aligned}$$

Therefore $I^W_{fg}(\tau) = 0$ if and only if $\Lambda_{fg}(z) = 0$ for all $z \in H_n(W, W\backslash V)$. Finally, observe that $g|_W$ is homotopic to a constant map. Therefore $f_* = 0$ if and only if $\Lambda_{fg}(z) = 0$ for all $z \in H_n(N, A)$ (Section 5 in [**30**]). □

Condition (*) ensures that only the primary obstruction to removability, i.e., the Lefschetz number, can be nonzero. Further investigation of necessary conditions of removability of coincidences will require computing higher order obstructions. The case when $f(F)$ is not a single point is best addressed in the context of Nielsen theory via Wecken type theorems [**33**]. In general, the homotopy of f cannot be always chosen arbitrarily small.

Especially important for the control theory applications are the following corollaries. They are applied to disappearance under perturbations of equilibria (Section 6) and controllability (Section 7) respectively.

Corollary 5.2. ***(Removability of fixed points)*** *Suppose the conditions of the theorem are satisfied for $N = M \times U$, where U is a manifold, $x \in M\backslash\partial M$ an isolated fixed point of $f : M \times U \to M$ (i.e., $f(x, u) = x$ for some $u \in U$), $F = \{x\} \times \{u \in U : g(x, u) = x\}$. Then there is a homotopy of f to a map f' such that f' has no fixed points in a neighborhood of F. The homotopy can be chosen arbitrarily small and constant on the compliment of a neighborhood of F.*

Proof. If $g : M \times U \to M$ is the projection then F is the coincidence set of f, g. □

Corollary 5.3. ***(Necessary condition of surjectivity)*** *Suppose the conditions of the theorem are satisfied for $F = f^{-1}(x)$ of $f : N \to M$. Then there is a homotopy of f to a map f' which is not onto; specifically, $x \notin f'(N)$. The homotopy can be chosen arbitrarily small and constant on the compliment of a neighborhood of F.*

Proof. If g is the constant map then F is the coincidence set of f, g. $\square$

These two corollaries are partial converses of Corollaries 4.4 and 4.5 respectively.

A submanifold F of N satisfies condition (*) if one of the following three conditions holds [**32**, Section 4]:

- *(a1) M is a surface, i.e., $n = 2$; or*
- *(a2) F is acyclic, i.e., $H_k(F) = 0$ for $k = 1, 2, \ldots$; or*
- *(a3) every component of F is a homology m-sphere, i.e., $H_k(F) = 0$ for $k \neq 0, m$, for the following values of m and n:*

  (1) $m = 4$ and $n \geq 6$;
  (2) $m = 5$ and $n \geq 7$;
  (3) $m = 12$ and $n = 7, 8, 9$, or *$n \geq 14$.*

6. Existence of equilibria.

In this section M is a compact orientable connected manifold with boundary ∂M, $\dim M = n$, U is a topological space.

A *discrete time control system* D_g is given by a map $g : M \times U \to M$, with U the *space of inputs*, M the *space of states* of the system.

We say that $D_{g'}$ is a *perturbation* of D_g if g' homotopic to g. To justify this definition recall that a system $D_{g'}$ is normally called a perturbation of D_g if g' is "close enough" to g in terms of the distance between $g(z)$ and $g'(z)$. However, if g' is a simplicial approximation of g [**2**, p. 251] then g and g' are homotopic. Thus we permit large but continuous perturbations of the system. Properties preserved under such perturbations may be called *strongly robust.*

As before suppose $\{a_1^k, \ldots, a_{m_k}^k\}$ is a basis for $H_k(M)$ and $\{x_1^k, \ldots, x_{m_k}^k\}$ the corresponding dual basis for $H^k(M)$.

Theorem 6.1. ***(Existence of equilibria)*** *If*

$$L(g_v) = (-1)^{ns} \sum_k (-1)^k \sum_j x_j^k \frown g_*(a_j^k \otimes v) \neq 0 \text{ for some } v \in H_s(U)$$

then every perturbation of the discrete time system D_g has an equilibrium.

Proof. In light of Corollary 4.4 we only need to show that the above formula for the Lefschetz class $L(g_v)$ of $g_v(x) = (-1)^{(n-i)s} g_*(x \otimes v)$, $x \in H_i(M)$, is true. Since the degree of g_v is s and $a_j^k \in H_k(M)$, we substitute $m = s$ and $i = k$ in Definition 4.1:

$$\begin{aligned} L(g_v) &= \sum_k (-1)^{k(k+s)} \sum_j x_j^k \frown (-1)^{(n-k)s} g_*(a_j^k \otimes v) \\ &= \sum_k (-1)^{k^2+ns} \sum_j x_j^k \frown g_*(a_j^k \otimes v), \end{aligned}$$

and the formula follows. $\square$

The following a generalization of a well known theorem about dynamical systems.

Corollary 6.2. *Suppose D_g is a perturbation of the constant system D_p, i.e., $p(x, u) = x$ for all u. If the Euler characteristic of M is nonzero, $\chi(M) \neq 0$, then D_g has an equilibrium.*

PROOF. Since $p_*(a_j^k \otimes v) = a_j^k$ if $v = 1$ and 0 otherwise, we have

$$\begin{aligned} L(g_v) &= \sum_k (-1)^k \sum_j x_j^k \frown p_*(a_j^k \otimes v) \\ &= \sum_k (-1)^k \sum_j 1 \\ &= \sum_k (-1)^k m_k \\ &= \chi(M). \end{aligned}$$

□

COROLLARY 6.3. *Suppose $M = \mathbf{S}^n$, and suppose one of the following conditions is satisfied:*

(1) $g_*(d \otimes 1) \neq (-1)^{n+1} d$, where d is the identity of $H_n(\mathbf{S}^n)$; or

(2) $g_*(1 \otimes v) \neq 0$ for some $v \in H_n(U)$.

Then every perturbation of the discrete time system D_g, $g : \mathbf{S}^n \times U \to \mathbf{S}^n$, has an equilibrium.

PROOF. Let's compute $L(g_v)$ for an arbitrary $v \in H_s(U)$. As $a_j^k \in H_k(M)$, we have $a_j^k \otimes v \in H_{k+s}(M \times U)$ and $g_*(a_j^k \otimes v) \in H_{k+s}(M)$. Since $H_i(M) = H_i(\mathbf{S}^n) = 0$ for all $i \neq 0, n$, we have $g_*(a_j^k \otimes v) = 0$ except for the following two cases. (1) Choose $v = 1 \in H_0(U), s = 0$, then either $k = 0, a_j^0 = 1, x_j^0 = 1$, or $k = n, a_j^n = d, x_j^n = \overline{d}$. (2) Choose $v \in H_n(U), s = n$, then $k = 0, a_j^0 = 1, x_j^0 = 1$. Here $\overline{d}$ is the dual of d, $\overline{d} \frown d = 1$. Thus we have

$$\begin{aligned} (1)\ L(g_1) &= (-1)^{n0}(1 \frown g_*(1 \otimes 1) + (-1)^n \overline{d} \frown g_*(d \otimes 1)) \\ &= 1 + (-1)^n \overline{d} \frown g_*(d \otimes 1); \\ (2)\ L(g_v) &= (-1)^{nn}(1 \frown g_*(1 \otimes v)) \\ &= (-1)^n g_*(1 \otimes v). \end{aligned}$$

Now, if either $L(g_1)$ or $L(g_v)$ is nonzero, then D_g has an equilibrium by the theorem.

□

Condition (1) means that the degree of $\overline{g}(\cdot) = g(\cdot, u_0) : \mathbf{S}^n \to \mathbf{S}^n$ is not equal to $(-1)^{n+1}$.

If $U = M$ is a compact Lie group and $g : M \times M \to M$ is the multiplication, then D_g has a equilibrium [**13**, Example 2.3]. For more examples, see [**13**], [**30**], [**31**].

In the control setting Corollary 5.2 reads as follows.

THEOREM 6.4. ***(Removability of equilibria)*** *Suppose U is a manifold and suppose $x \in M \backslash \partial M$ is an isolated equilibrium of D_g. Suppose condition (*) is satisfied for $F = \{x\} \times \{u \in U : g(x, u) = x\}$ and*

$$f_* : H_n(W, W \backslash V) \to H_n(D, D \backslash B) = \mathbf{Q} \text{ is zero,}$$

where $V \subset \overline{V} \subset W$ and $B \subset \overline{B} \subset D \subset M \backslash \partial M$ are neighborhoods of F and x respectively. Then this equilibrium can be removed by an arbitrarily small perturbation restricted to a neighborhood of F.

7. Controllability.

In this section M is a compact orientable connected manifold with boundary ∂M, $\dim M = n$, U is a topological space.

Suppose a discrete system D_f is given by $f : M \times U \to M$. The system D_f is called *controllable* [**34**] if any state can be reached from any other state by means of f, i.e., for each pair of states $x, y \in M$ there are inputs $u_0, ..., u_r \in U$ such that $x_1 = f(u_0, x), x_2 = f(u_1, x_1), ..., y = x_{r+1} = f(u_r, x_r)$, notation $x \rightsquigarrow_f y$.

Below this notion is generalized in three, nontypical but topologically appropriate, ways. First, we consider the possibility of an arbitrary state reached not from any given state but from a state in a particular subset L of M. Second, as before we permit arbitrary, not necessarily small, perturbations of f. Third, instead of looking into controllability of a new, perturbed, system D_g, where g is homotopic to f, we allow for consecutive applications of possibly different maps each homotopic to f.

Definition 7.1. Given $L \subset M$, let $f' : L \times U \to M$ be the restriction of f. Then the system is called *strongly robustly controllable from* L if for any map f_0 homotopic to f', any maps $f_1, ..., f_r$ homotopic to f, and for each $y \in M$ there is $x \in L$ and inputs $u_0, ..., u_r \in U$ such that

$$x_1 = f_0(x, u_0), x_2 = f_1(x_1, u_1), ..., y = x_{r+1} = f_r(x_r, u_r).$$

Then the system is controllable if it controllable from any point.

It is clear that controllability is equivalent to surjectivity of several iterations of f. To deal with surjectivity we apply Corollary 4.5 which requires f to be a map of pairs. For this purpose in this section we make the following assumption about D_f. If the initial state lies at the boundary ∂M of M then the next state, regardless of the input, lies within a certain neighborhood of ∂M. For simplicity we make a topologically equivalent assumption,

$$f(\partial M \times U) \subset \partial M.$$

Next, let U' be the set of controls that take any given state to the boundary of M, i.e.,

$$U' = \{u \in U : f(x, u) \in \partial M \text{ for all } x \in M\}.$$

Then $f(M \times U') \subset \partial M$. Combining this with the above assumption we conclude that f is a map of pairs, $f : (M, \partial M) \times (U, U') \to (M, \partial M)$. Let $L' = L \cap \partial M$, then $f' : (L, L') \times (U, U') \to (M, \partial M)$ is also a map of pairs.

The following theorem translates the above "reachability" condition into the language of homology: any element of $H_n(M, \partial M) = \mathbf{Q}$ can be reached from some $a_0 \in H_*(L, L')$ by means of f_*.

Theorem 7.2. ***(Sufficient condition of robust controllability)*** *Suppose that there are $a_0 \in H_p(L, L')$, $v_0 \in H_{s_0}(U, U'), ..., v_r \in H_{s_r}(U, U')$ such that*

$$a_1 = f'_*(a_0 \otimes v_0), a_2 = f_*(a_1 \otimes v_1), ..., a_{r+1} = f_*(a_r \otimes v_r) \in H_n(M, \partial M) \backslash \{0\}.$$

Then the discrete time system D_f is strongly robustly controllable from L.

Here, if $a_i \in H_{n_i}(M, \partial M), i = 0, 1, 2, ..., r$, then $n_0 = p, n_1 = p + s_0, n_2 = n_1 + s_1, ..., n_{r+1} = n_r + s_r = n$. Thus we have a sequence of homology classes $a_0, .., a_r$ of $(M, \partial M)$ "climbing" dimensions from p to n.

PROOF. The result of consecutive applications of f is defined as a map $F : (L,L') \times (U,U')^{r+1} \to (M,\partial M)$ given by

$$F(x,u_0,...,u_r) = f(...f(f'(x,u_0),u_1),...,u_r),$$

i.e., it is given by the composition

$$F : (L,L') \times (U,U') \times ... \times (U,U') \xrightarrow{f' \times Id}$$
$$(M,\partial M) \times (U,U') \times ... \times (U,U') \xrightarrow{f \times Id} ....$$

Then $x \rightsquigarrow_f F(x,u_0,...,u_r)$. Suppose a map f_0 is homotopic to f' and maps $f_1,...,f_r$ are homotopic to f. The result of consecutive applications of $f_0,...,f_r$ is defined as a map $G : (L,L') \times (U,U')^{r+1} \to (M,\partial M)$ given by

$$G(x,u_0,...,u_r) = f_r(...f_1(f_0(x,u_0),u_1),...,u_r).$$

Therefore strong robust controllability from L means that $G : L \times U^{r+1} \to M$ is onto. By Corollary 4.5 if

$$F_* : H_n((L,L') \times (U,U') \times ... \times (U,U')) \to H_n(M,\partial M) = \mathbf{Q}$$

is nonzero then every map homotopic to F is onto. Since G is clearly homotopic to F, all we need to prove is that F_* is nonzero. By the Künneth theorem F_* is given by the composition

$$F_* : H_*(L,L') \otimes H_*(U,U') \otimes ... \otimes H_*(U,U') \xrightarrow{f'_* \otimes Id}$$
$$H_*(M,\partial M) \otimes H_*(U,U') \otimes ... \otimes H_*(U,U') \xrightarrow{f_* \otimes Id} ....$$

Now the condition of the theorem implies that $f_*(...f_*(f'_*(a_0 \otimes v_0) \otimes v_2) \otimes ... \otimes v_r) \neq 0$ for some $a_0 \in H_p(L,L')$ and some $v_0 \in H_{s_1}(U,U'),...,v_r \in H_{s_r}(U,U')$ such that $p + s_1 + ... + s_r = n$. Therefore $F_*(a_0 \otimes v_0 \otimes v_2 \otimes ... \otimes v_r) \neq 0$. $\square$

Moreover, it is clear that what we have is the "finite time reachability", i.e., every state can be reached in a finite number, $r+1$, of steps and that number is common for all states.

The theorem involves multiple iterations of f_* while it is preferable to have a condition involving only f_* itself. Let's consider a case when this is possible.

Consider first a simple example, $U = \mathbf{S}^1$, $U' = \varnothing$, $M = \mathbf{T}^n = (\mathbf{S}^1)^n$, and $f : \mathbf{S}^1 \times \mathbf{T}^n \to \mathbf{T}^n$ is given by $f(u,x_1,...,x_n) = (u,x_1,...,x_{n-1})$. This may serve as a model for a robotic arm with n joints where only the first joint can be controlled directly and the next state of a joint is "read" from the current state of the previous joint. The system is obviously controllable. Indeed after n iterations with inputs $u_1,...,u_n$ the system's state is $(u_n,...,u_1)$. Whether the system is robustly controllable is not as obvious. The affirmative answer is provided by the theorem as follows. Let L be a point, $p = 0$. Now, with d the identity of $H_1(\mathbf{S}^1)$ we choose

$$\begin{aligned}
&v_0 = v_1 = ... = v_n = d \in H_1(\mathbf{S}^1), \text{ and} \\
&a_0 = 1 \in H_0(\mathbf{T}^n), \\
&a_1 = d \in H_1(\mathbf{T}^n), \\
&a_2 = d \otimes d \in H_2(\mathbf{T}^n), \\
&\quad ... \\
&a_n = d \otimes ... \otimes d \in H_n(\mathbf{T}^n).
\end{aligned}$$

More generally, suppose the state space M has the product structure, $M = K_1 \times ... \times K_s$, where K_i are manifolds of dimensions k_i. Suppose $f = (h_1, ..., h_s)$, where $h_i : U \times M \to K_i$. For $i = 1, ..., s$, define maps $h_i^a : K_{i-1} \to K_i$, where $K_0 = U$, by $h_i^a(x_{i-1}) = h_i(a_0, ..., a_{i-2}, x_{i-1}, a_i, ..., a_s)$. If all h_i^a are onto then the system is controllable. According to Corollary 4.5 it suffices to require that all $h_{i*}^a : H_{k_i}(K_{i-1}) \to H_{k_i}(K_i)$ are nonzero, $i = 1, ..., s$.

The above theorem can be informally understood as follows. If there are some submanifolds $M_1, ..., M_r$, $\dim M_i = n_i$, of M such that $M_0 = L, M_1 = f(M_0 \times U)$, $M_2 = f(M_1 \times U), ..., M = f(M_r \times U)$ then the system is controllable. It means that the restrictions $f_0 : L \times U \to M_1, f_1 : M_1 \times U \to M_2, ..., f_r : M_r \times U \to M$ of f are surjective. This holds provided $f_{i*}(O_{M_i} \otimes O_U) = q_i O_{M_{i+1}}$, where $O_{M_i} \in H_{n_i}(M_i)$ is the fundamental class of M_i, for some $q_i \in \mathbf{Q}$. Since each O_{M_i} corresponds to $a_i = J_{i*}(O_{M_i}) \in H_{n_i}(M)$, where $J_i : M_i \to M$ is the inclusion, we arrive at the requirement of the theorem. The robustness of each of these surjectivity conditions can be tested by means of Corollary 5.3. As a special case we have the following.

Theorem 7.3. ***(Necessary condition of robust controllability)*** *Suppose U is a manifold and there is a fiber $F = f^{-1}(x), x \in M$, of f satisfying condition (*) and*

$$f_* : H_n(W, W \backslash V) \to H_n(D, D \backslash B) = \mathbf{Q} \text{ is zero,}$$

where $V \subset \overline{V} \subset W$ and $B \subset \overline{B} \subset D \subset M \backslash \partial M$ are neighborhoods of F and x respectively. Then there is an arbitrarily small perturbation restricted to a neighborhood of F of the system D_f which is not controllable from M; specifically, x is unreachable from any point.

Proof. Corollary 5.3 implies that there is g homotopic to f such that $x \notin g(M \times U)$. $\square$

8. Continuous systems.

In this section we outline, in less details than above, the possibilities of applying Lefschetz numbers to continuous systems.

In this section M is a compact orientable connected smooth manifold with boundary ∂M, $\dim M = n$. Let TM be the tangent bundle of M, then $\dim TM = 2n$.

A *continuous time control system* C_h [**27**, p. 16] is defined as a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{h} & TM, \\ \downarrow^{p} & \swarrow^{\pi_M} & \\ M & & \end{array}$$

where $p : Q \to M$ is a fiber bundle over M and π_M is the projection. Thus C_h is a parametrized vector field on M.

We say that $x \in M$ is an *equilibrium* of this system if there is $y \in Q$ such that $h(y) = (x, 0) \in TM, x = p(y) \in M$. Detecting an equilibrium can be restated as a coincidence problem. Suppose $i : M \to TM$ is the inclusion and $p_1 : Q \times M \to Q$, $p_2 : Q \times M \to M$ are the projections. Define the maps $f, g : Q \times M \to TM$ by $f = hp_1, g = ip_2$. Then a coincidence of the pair f, g is an equilibrium of the system C_h. Therefore equilibria can be detected by means of the coincidence results in Section 4 and their robustness can be studied by means of the results of Section 5.

We have a simpler coincidence problem when M is parallelizable, i.e., TM is isomorphic to $M \times \mathbf{R}^n$. For example, $\mathbf{S}^1$, $\mathbf{S}^3$, $\mathbf{S}^7$ are parallelizable. Let $q : TM \simeq M \times \mathbf{R}^n \to M$ be the projection. Then a coincidence of the pair qh, p is an equilibrium of the system C_h and we can use Theorem 4.3 to detect equilibria and Theorem 5.1 to study their robustness. In fact D_{qh} is a discrete control system associated with the continuous system C_h. In particular, when $Q = M \times U$, the results of Sections 6 and 7 can be applied to study equilibria and controllability of C_h.

For a general M a discrete system D_f associated to the continuous system C_h may be constructed as follows.

Let $\mathcal{A}$ be the topological space of *admissible controls* associated with C_h, i.e., a set of functions $z : [0, d] \to Q$, for all $d \in \mathbf{R}$. A map $c_z : [0, d] \to M$ is called a *trajectory* of the control system if there exists a control $z \in \mathcal{A}$ satisfying: $pz = c_z$ and $\dfrac{d}{dt} c_z = hz$.

We assume that $Q = M \times U$, where U is the topological space of all possible inputs, and $p : Q = M \times U \to M$ is the projection. Then $\mathcal{A}$ is the set of pairs (c, p), where $c : [0, d] \to M$ is a trajectory and $p : [0, d] \to U$ is a function representing the input. To simplify things even further we consider only constant inputs. First we assume that the system C_h satisfies the following existence and uniqueness property: for every $x \in M$ and any *constant* input $p(t) = u \in U$ there is a unique trajectory c such that $c(0) = x$ and $(c, p) \in \mathcal{A}$. Then the following end point map $f_d : M \times U \to M$ is well defined. We let $f_d(x, u) = c(d)$, where $c : [0, d] \to M$ is the above trajectory. Assume also that the map $f = f_d$ is continuous. Then for each $d \geq 0$ we have a discrete time control system D_f.

Next, the system C_h is called *controllable* if any state can be reached from any other state, i.e., for each pair of states $x, y \in M$ there is a trajectory $c : [0, d] \to M$ such that $x = c(0), y = c(d)$.

We make the same assumption about D_f as in Section 7: if the initial state lies at the boundary ∂M of M then the next state, regardless of the input, lies within a certain neighborhood W of ∂M, or, alternatively, $f(\partial M \times U) \subset \partial M$. In particular, this condition is satisfied if $h(x, u)$ is tangent to ∂M for all $x \in \partial M$. Let U' be the set of controls that take any given state to the the boundary ∂M, i.e.,

$$U' = \{u \in U : f(x, u) \in \partial M \text{ for all } x \in M\}.$$

Then f is a map of pairs, $f : (M, \partial M) \times (U, U') \to (M, \partial M)$. Given a subset L of M, let $L' = L \cap \partial M$ and let $f' : (L, L') \times (U, U') \to (M, \partial M)$ be the restriction of f.

Theorem 8.1. ***(Sufficient condition of controllability)*** *Suppose that there are $a_0 \in H_p(L, L')$, $v_0 \in H_{s_0}(U, U'), ..., v_r \in H_{s_r}(U, U')$ such that*

$$a_1 = f'_*(a_0 \otimes v_0), a_2 = f_*(a_1 \otimes v_1), ..., a_{r+1} = f_*(a_r \otimes v_r) \neq 0.$$

Then the continuous time system C_h is controllable from L by means of piece-wise constant controls.

Proof. The discrete system D_f is controllable from L by Theorem 7.2. $\square$

It follows also that if for a small enough $\varepsilon > 0$ a map $k : Q \to TM$ satisfies $d(k(z), h(z)) < \varepsilon$ for all $z \in Q$, where d is the distance on TM, and the system C_k

satisfies all of the above assumptions, then C_k is also controllable. We can say then that C_h is *robustly controllable*.

Consider the applicability of this theorem to local controllability or controllability in a Euclidean space. In either case M is the n-ball. Then $H_i(M, \partial M)$ is nontrivial only in dimension n. As a result the above "chain" of homology classes $a_1, a_2, ..., a_{r+1}$ has to have only one "link", $a_1 = f'_*(a_0 \otimes v_0) \in H_n(M, \partial M) \backslash \{0\}$. Thus the theorem reduces to the claim of one-step controllability provided f'_{*n} is nonzero. As a result the similarity between the homology reachability condition of the theorem and the Lie bracket condition [**27**, Section 3.1] does not materialize. I believe however that a generalization of Theorem 7.2 will provide a necessary connection.

Observe also that if $\partial M = \varnothing$, then $f = f_d$ is homotopic to the constant map f_0 under the homotopy $H(t, x, u) = f_t(x, u)$, hence $f_* = 0$. Therefore the condition of the theorem is never satisfied.

Here's another approach to controllability. Let $\mathcal{A}'$ be the set of controls whose trajectories have one of the end points at the boundary of M, i.e.,

$$\mathcal{A}' = \{z : [0, d] \to Q, z \in \mathcal{A}, c_z(0) \in \partial M \text{ or } c_z(d) \in \partial M\}.$$

Define $G(u) = (c_z(0), c_z(d))$, the end points of the trajectory $c_z = pz : [0, d] \to M$ corresponding to z. Then $G : (\mathcal{A}, \mathcal{A}') \to (M \times M, \partial(M \times M))$ is a well defined map of pairs.

Theorem 8.2. ***(Sufficient condition of controllability)*** *If*

$$G_* : H_{2n}(\mathcal{A}, \mathcal{A}') \to H_{2n}(M \times M, \partial(M \times M)) = \mathbf{Q} \text{ is non-zero}$$

then the continuous time system C_h is controllable.

Proof. By Corollary 4.5 G is onto. □

A similar condition is found in [**28**], where a boundary operator $l : AC([0, 1], \mathbf{R}^n) \times L^\infty([0, 1], \mathbf{R}^n) \to \mathbf{R}^p$ is considered instead of G. One of the conditions of controllability is $\deg l_0 \neq 0$, where l_0 is the restriction of l to some p-dimensional subspace and $\deg l_0$ its topological degree.

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